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Dissipative hierarchies and resonance solitons for KP-II and MKP-II

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Abstract

We show that dissipative solitons (dissipatons) of the second and the third members of SL(2,R) AKNS hierarchy give rise to the real solitons of KP-II, while for SL(2,R) Kaup-Newell hierarchy they give solitons of MKP-II. By the Hirota bilinear form for both flows, we find new bilinear system for these equations, and one- and two-soliton solutions. For special values of parameters our solutions show resonance behaviour with creation of four virtual solitons. We first time created four virtual soliton resonance solution for KP-II and established relations of it with degenerate four-soliton solution in the Hirota-Satsuma bilinear form for KP-II. Our approach allows one to interpret the resonance soliton as a composite object of two dissipative solitons in $1+1$ dimensions.

Keywords: Dissipative hierarchies; Resonance solitons; Kadomtsev–Petviashvili equation; Hirota Method

1. Introduction

Recently, "dissipative" version of AKNS hierarchy has been considered in connection with $1+1$ dimensional (lineal) gravity models [12].The gauge theoretical formulation of these models is based on the Cartan–Einstein vielbein or the moving frame method [7]. In terms of these variables in $1+1$ dimensions one deals with so called BF gauge theory and the zero curvature equations of motion, providing a link with soliton equations [12]. But in these variables, as the "square root" of the pseudo-Riemannian metric, the soliton equations have dissipative form, this is why we called them the dissipative solitons or dissipatons [12]. The second-order flow of the hierarchy, the dissipative version of the nonlinear Schrodinger equation which we called the reaction–diffusion system (RD) [12,14,15], is a couple of nonlinear heat and anti-heat equations, so that the corresponding system is conservative. It has a rich resonance dynamics [14,15]. The dissipaton of that system is the pair of two real functions, one of which is exponentially growing, and another decaying, in the space and time. But the product of these two functions has the perfect soliton form. From another site, dissipative version of the derivative nonlinear Schrodinger equations (DNLS) from SL(2,R) Kaup-Newell (KN) hierarchy also admits dissipative soliton solutions with the resonance interaction [14,15]. Moreover, these resonances show the chirality properties, propagating only in one direction. The problem is to find the real soliton equation for the product of dissipatons.

In the present paper we study resonance dissipative solitons in AKNS and the Kaup-Newell hierarchies and show that they give rise to the real solitons of $2+1$ dimensional KP-II and MKP-II equations correspondingly. Our approach

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is based on a new method to generate solutions of these equations: we show that if one considers a simultaneous solution of the second and the third flows from the AKNS and KN hierarchies, then the product e^+e^- satisfies the KP-II and MKP-II equations. Using these results we construct new bilinear representation of KP-II and MKP-II equations with one- and two-soliton solutions. We show that our KP-II two-soliton solution corresponds to the degenerate four-soliton solution in the standard Hirota form of KP , and displays the four virtual soliton resonance. For MKP-II we show that the chirality property of dissipatons poses restriction on the soliton collision angles.

2. KP-II resonance solitons

2.1. KP-II and SL(2,R) AKNS hierarchy

The dissipative SL(2,R) AKNS hierarchy of evolution equations [1] with times $t_0, t_1, t_2, \ldots, t_N, \ldots$, for real functions $e^+(x, t_N)$, $e^-(x, t_N)$,

$$\sigma_3 \begin{pmatrix} e^+ \\ e^- \end{pmatrix}_{t_N} = \Re^{N+1} \begin{pmatrix} e^+ \\ e^- \end{pmatrix}, \tag{1}$$

where $N = 0, 1, 2, \ldots$ $(\Lambda < 0)$, is generated by the recursion operator $\Re$

$$\Re = \begin{pmatrix} \partial_x + \frac{\Lambda}{4} e^+ \int^x e^- & -\frac{\Lambda}{4} e^+ \int^x e^+ \\ +\frac{\Lambda}{4} e^- \int^x e^- & -\partial_x - \frac{\Lambda}{4} e^- \int^x e^+ \end{pmatrix}. \tag{2}$$

Then, the second and third members of AKNS hierarchy appear as

$$\begin{cases} e^+_{t_1} = e^+_{xx} + \frac{\Lambda}{4} e^+ e^- e^+ \\ -e^-_{t_1} = e^-_{xx} + \frac{\Lambda}{4} e^+ e^- e^- \end{cases} \tag{3}$$

and

$$\begin{cases} e^+_{t_2} = e^+_{xxx} + \frac{3\Lambda}{4} e^+ e^- e^+_x \\ e^-_{t_2} = e^-_{xxx} + \frac{3\Lambda}{4} e^+ e^- e^-_x \end{cases} \tag{4}$$

respectively. The first system (3), the dissipative version of the Nonlinear Schrödinger equation, is called the reaction–diffusion (RD) system [12]. It is connected with gauge theoretical formulation of $1 + 1$ dimensional gravity, the constant curvature surfaces in pseudo-Euclidean space [12] and the NLS soliton problem in the quantum potential [12,14,15].

AKNS hierarchy allows us to develop a method to find solution for $(2 + 1)$ dimensional Kadomtsev-Petviashvili (KP) equation. Depending on sign of dispersion, two types of the KP equations are known. The minus sign in the right side of the KP corresponds to the case of negative dispersion and called KP-II. To relate KP-II with AKNS hierarchy let us consider the pair of functions $e^+(x, y, t)$, $e^-(x, y, t)$ satisfying the second and the third members of the dissipative AKNS hierarchy. Here we renamed time variables t_1 as y and t_2 as t. Differentiating according to t and y, Eqs. (3) and (4) correspondingly, we can see that they are compatible.

Proposition 2.1.1. *Let the functions* $e^+(x, y, t)$ *and* $e^-(x, y, t)$ *are solutions of* Eqs. (3) and (4) *simultaneously. Then the function* $U(x, y, t) \equiv e^+e^-$ *satisfies the Kadomtsev-Petviashivili* (*KP-II*) *equation*

$$\left(4U_t + \frac{3\Lambda}{4}(U^2)_x + U_{xxx}\right)_x = -3U_{yy}. \tag{5}$$

The proof is straigthforward.[1]

[1] Similar results are known also as the symmetry reductions of KP [9,4,3].

2.2. Bilinear representation of KP-II by AKNS flows

Using bilinear representations for systems (3) and (4) [14,15] and Proposition 2.1.1 we can find bilinear representation for KP-II. We consider $G^{\pm}$ and F as real functions of three variables $G^{(\pm)} = G^{(\pm)}(x, y, t)$, $F = F(x, y, t)$, and require for these functions to be a solution of corresponding bilinear systems for that equations simultaneously. Since the second equation in both systems is the same, it is sufficient to consider the next bilinear system

$$\begin{cases} (\pm D_y - D_x^2)(G^{\pm} \cdot F) = 0 \\ (D_t + D_x^3)(G^{\pm} \cdot F) = 0 \\ D_x^2(F \cdot F) = -2G^+G^- \end{cases} \tag{6}$$

Then, according to Proposition 2.1.1, any solution of this system generates a solution of KP-II. From the last equation we can derive U directly in terms of function F only

$$U = e^+e^- = \frac{8}{-\Lambda}\frac{G^+G^-}{F^2} = \frac{4}{\Lambda}\frac{D_x^2(F \cdot F)}{F^2} = \frac{8}{\Lambda}\frac{\partial^2}{\partial x^2} \ln F \tag{7}$$

Simplest solution of this system

$$G^{\pm} = \pm e^{\eta_1^{\pm}}, \qquad F = 1 + \frac{e^{(\eta_1^+ + \eta_1^-)}}{(k_1^+ + k_1^-)^2}, \tag{8}$$

where $\eta_1^{\pm} = k_1^{\pm}x \pm (k_1^{\pm})^2 y - (k_1^{\pm})^3 t + \eta_1^{\pm(0)}$, defines one-soliton solution of KP-II according to Eq. (7)

$$U = \frac{2(k_1^+ + k_1^-)^2}{\Lambda \cosh^2(1/2)[(k_1^+ + k_1^-)x + (k_1^{+2} - k_1^{-2})y - (k_1^{+3} + k_1^{-3})t + \gamma]}, \tag{9}$$

where $\gamma = -\ln(k_1^+ + k_1^-)^2 + \eta_1^{+(0)} + \eta_1^{-(0)}$. This soliton is a planar wave wall travelling in an arbitrary direction and called the line soliton.

2.3. Two-soliton solution

Continuing Hirota's perturbation we find two-soliton solution in the form

$$G^{\pm} = \pm(e^{\eta_1^{\pm}} + e^{\eta_2^{\pm}} + \alpha_1^{\pm}e^{\eta_1^+ + \eta_1^- + \eta_2^{\pm}} + \alpha_2^{\pm}e^{\eta_2^+ + \eta_2^- + \eta_1^{\pm}}), \tag{10}$$

$$F = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2} + \frac{e^{\eta_1^+ + \eta_2^-}}{(k_{12}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2} + \beta e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-}, \tag{11}$$

where $\eta_i^{\pm} = k_i^{\pm}x \pm (k_i^{\pm})^2 y - (k_i^{\pm})^3 t + \eta_i^{\pm(0)}$, $\quad k_{ij}^{ab} = k_i^a + k_j^b$, $\quad (i, j = 1, 2), (a, b = +-)$,

$$\alpha_1^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{11}^{+-}k_{21}^{\pm\mp})^2}, \quad \alpha_2^{\pm} = \frac{(k_1^{\pm} - k_2^{\pm})^2}{(k_{22}^{+-}k_{12}^{\pm\mp})^2}, \quad \beta = \frac{(k_1^+ - k_2^+)^2(k_1^- - k_2^-)^2}{(k_{11}^{+-}k_{12}^{+-}k_{21}^{+-}k_{22}^{+-})^2}.$$

Then, it provides two-soliton solution of KP-II according to Eq. (7).

2.4. Degenerate four-soliton solution

However for KP-II another bilinear form in terms of function F only is known [5]

$$(D_xD_t + D_x^4 + D_y^2)(F \cdot F) = 0 \tag{12}$$

Thus, it is natural to compare soliton solutions of our bilinear Eq. (6) with the ones given by this equation. To solve Eq. (12) we consider $F = 1 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots$. The solution $F_1 = e^{\eta_1}$, where $\eta_1 = k_1x + \Omega_1 y + \omega_1 t + \eta_1^0$, and dispersion $k_1\omega_1 + k_1^4 + \Omega_1^2 = 0$ with $F_n = 0$, $(n = 2, 3, \ldots)$, under identification $k_1 = k_1^+ + k_1^-$, $\Omega_1 = \sqrt{3}(k_1^{+2} - k_1^{-2})$, $\omega_1 = -4(k_1^{+3} + k_1^{-3})$, and rescaling $4t \to t$, $\sqrt{3}y \to y$, determines one-soliton solution of KP-II (5). We realize that it coincides with our one-soliton solution (9). But two-soliton solution of Eq. (12) [13] does not correspond to our two-soliton solution (10)and (11). Appearance of four different terms $e^{\eta_i^\pm+\eta_k^\pm}$ in Eq. (11), suggests that our two-soliton solution should correspond to some degenerate case of four-soliton solution of Eq. (12). To construct four-soliton solution first we find following solutions of bilinear Eq. (12)

$$F_1 = e^{\eta_1}, \quad F_2 = e^{\eta_2}, \quad F_4 = e^{\eta_3}, \tag{13}$$

where $\eta_i = k_i x + \Omega_i y + \omega_i t + \eta_i^0, \quad i = 1, 2, 3,$ dispersion

$$k_i\omega_i + k_i^4 + \Omega_i^2 = 0 \tag{14}$$

and

$$F_3 = \alpha_{12}e^{\eta_1+\eta_2}, \quad F_5 = \alpha_{13}e^{\eta_1+\eta_3}, \quad F_6 = \alpha_{23}e^{\eta_1+\eta_3}, \tag{15}$$

where

$$\alpha_{ij} = -\frac{(k_i - k_j)(\omega_i - \omega_j) + (k_i - k_j)^4 + (\Omega_i - \Omega_j)^2}{(k_i + k_j)(\omega_i + \omega_j) + (k_i + k_j)^4 + (\Omega_i + \Omega_j)^2}, \quad i, j = 1, 2, 3. \tag{16}$$

Then we parameterize our solution in the form

$$\begin{aligned}
k_1 &= k_1^+ + k_1^-, \quad \omega_1 = -4(k_1^{+3} + k_1^{-3}), \quad \Omega_1 = \sqrt{3}(k_1^{+2} - k_1^{-2}),\\
k_2 &= k_2^+ + k_2^-, \quad \omega_2 = -4(k_2^{+3} + k_2^{-3}), \quad \Omega_2 = \sqrt{3}(k_2^{+2} - k_2^{-2}),\\
k_3 &= k_1^+ + k_2^-, \quad \omega_3 = -4(k_1^{+3} + k_2^{-3}), \quad \Omega_3 = \sqrt{3}(k_1^{+2} - k_2^{-2}),\\
k_4 &= k_2^+ + k_1^-, \quad \omega_4 = -4(k_2^{+3} + k_1^{-3}), \quad \Omega_4 = \sqrt{3}(k_2^{+2} + k_1^{-2}),
\end{aligned} \tag{17}$$

satisfying dispersion relations (14). Substituting these parameterizations to above solutions we find that

$$\alpha_{13} = 0 \Rightarrow F_5 = 0, \quad \alpha_{23} = 0 \Rightarrow F_6 = 0 \tag{18}$$

continuing Hirota's perturbation with solution $F_7 = e^{\eta_4}$, where $\eta_4 = k_4x + \Omega_4 y + \omega_4 t + \eta_4^0$, we find that $F_8 = \alpha_{14}e^{\eta_1+\eta_4}$, where

$$\alpha_{14} = -\frac{(k_1 - k_4)(\omega_1 - \omega_4) + (k_1 - k_4)^4 + (\Omega_1 - \Omega_4)^2}{(k_1 + k_4)(\omega_1 + \omega_4) + (k_1 + k_4)^4 + (\Omega_1 + \Omega_4)^2} \tag{19}$$

and after the parameterizations given above (17) it also vanishes

$$\alpha_{14} = 0 \Rightarrow F_8 = 0 \tag{20}$$

The next solution $F_9 = \alpha_{24}e^{\eta_2+\eta_4}$, where

$$\alpha_{24} = -\frac{(k_2 - k_4)(\omega_2 - \omega_4) + (k_2 - k_4)^4 + (\Omega_2 - \Omega_4)^2}{(k_2 + k_4)(\omega_2 + \omega_4) + (k_2 + k_4)^4 + (\Omega_2 + \Omega_4)^2}, \tag{21}$$

also is zero

$$\alpha_{24} = 0 \Rightarrow F_9 = 0. \tag{22}$$

Then we have $F_{10} = 0$ and $F_{11} = \alpha_{34}e^{\eta_3+\eta_4}$, where

$$\alpha_{34} = -\frac{(k_3 - k_4)(\omega_3 - \omega_4) + (k_3 - k_4)^4 + (\Omega_3 - \Omega_4)^2}{(k_3 + k_4)(\omega_3 + \omega_4) + (k_3 + k_4)^4 + (\Omega_3 + \Omega_4)^2}. \tag{23}$$

When it is checked for higher order terms we find that $F_{12} = F_{13} = \cdots = 0$. Thus, we have degenerate four-soliton solution of Eq. (12) in the form

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_4} + \alpha_{12}e^{\eta_1+\eta_2} + \alpha_{34}e^{\eta_3+\eta_4} \tag{24}$$

Comparing this solution with the one in Eq. (11) and taking into account that according parameterizations (17), $\eta_1 + \eta_2 = \eta_3 + \eta_4$, we see that they coincide. The above consideration shows that our two-soliton solution of KP-II corresponds to the degenerate four-soliton solution in the canonical Hirota form (12). Moreover, it allows us to find new four virtual soliton resonance for KP-II.

2.5. Resonance interaction of planar solitons

Choosing different values of parameters for our two-soliton solution we find resonance character of soliton's interaction. For the next choice of parameters $k_1^+ = 2$, $k_1^- = 1$, $k_2^+ = 1.5$, $k_2^- = 0.5$, and vanishing value of the position shift constants, we obtained two-soliton solution moving in the plane with constant velocity, with creation of four, so called virtual solitons (solitons without asymptotic states at infinity) (Fig. 1a and b).

The resonance character of our planar soliton interactions is related with resonance nature of dissipatons in 1+1 AKNS hierarchy. It has been reported also in several systems, but the four virtual soliton resonance does not seem to have been done for KP-II [6] prior to our work. Recently we realized that resonance solitons for KP-II have been constructed independently also by Biondini and Kodama [2,8] using Sato's theory. Then, the comparision shows that our bilinear constraint plays the similar role as the Toda lattice in their paper.

3. MKP-II resonance solitons

3.1. MKP-II and Kaup-Newell hierarchy

The KN hierarchy for functions $q(x, t_N)$, $r(x, t_N)$ has the form [16]

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_N} = JL^N \begin{pmatrix} q \\ r \end{pmatrix} \tag{25}$$

where the operator

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{26}$$

is the first symplectic form, while

$$L = \frac{1}{2} \begin{pmatrix} -\partial_x - r \int^x q\partial_x & -r \int^x r\partial_x \\ -q \int^x q\partial_x & \partial_x - q \int^x r\partial_x \end{pmatrix} \tag{27}$$

is the recursion operator of the hierarchy. The second flow of the hierarchy is the system

$$q_{t_2} = \frac{1}{2}[q_{xx} + (q^2 r)_x], \tag{28a}$$

$$r_{t_2} = \frac{1}{2}[-r_{xx} + (r^2 q)_x], \tag{28b}$$

while the third one is

$$q_{t_3} = -\frac{1}{4}[q_{xx} + 3rqq_x + \frac{3}{2}(r^2q^2)q]_x, \tag{29a}$$

$$r_{t_3} = -\frac{1}{4}[r_{xx} - 3rqr_x + \frac{3}{2}(r^2q^2)r]_x, \tag{29b}$$

For the SL(2,R) case of KN hierarchy we have real time variables t_2, t_3 which we denote as $y \equiv t_2/2$, and $t \equiv -t_3/4$. In this case functions q and r are real, and we denote them as

$$e^+ \equiv q, \quad e^- \equiv -r \tag{30}$$

Then we have the DRD system [11]

$$e_y^+ = e_{xx}^+ - (e^+e^-e^+)_x, \tag{31a}$$

$$e_y^- = -e_{xx}^- - (e^+e^-e^-)_x, \tag{31b}$$

and

$$e_t^+ = e_{xxx}^+ - 3(e^+e^-e_x^+)_x + \frac{3}{2}((e^+e^-)^2e^+)_x, \tag{32a}$$

$$e_t^- = e_{xxx}^- + 3(e^+e^-e_x^-)_x + \frac{3}{2}((e^+e^-)^2e^-)_x, \tag{32b}$$

Now we consider the pair of functions of three variables $e^+(x, y, t)$ and $e^-(x, y, t)$ satisfying the systems (31) and (32). These systems are compatible since they belong to the same hierarchy for different times. This can be also checked directly from compatibility condition $e_{ty}^\pm = e_{yt}^\pm$ by using following conservation laws for Eqs. (31) and (32), respectively

$$(e^+e^-)_y = \left[(e_x^+e^- - e^+e_x^-) - \frac{3}{2}(e^+e^-)^2\right]_x, \tag{33}$$

$$(e^+e^-)_t = [(e^+e^-)_{xx} - 3(e_x^+e_x^-) + 3(e^+e^-)(e^+e_x^- - e_x^+e^-) + \frac{5}{2}(e^+e^-)^3]_x$$

Proposition 3.1.1. *Let the functions $e^+(x, y, t)$ and $e^-(x, y, t)$, are solutions of the systems (31) and (32) simultaneously. Then , the function $U(x, y, t) \equiv e^+e^-$ satisfies the modified Kadomtsev-Petviashvili equation (MKP-II)*

$$\left(-4U_t + U_{xxx} - \frac{3}{2}U^2U_x - 3U_x\partial_x^{-1}U_y\right)_x = -3U_{yy} \tag{34}$$

or written in another form

$$-4U_t + U_{xxx} - \frac{3}{2}U^2U_x - 3U_xW = -3W_y \tag{35a}$$

$$W_x = U_y \tag{35b}$$

(The second form appears from the first one by introducing auxiliary variable W according to Eq. (35b) and integration in variable x.)

3.2. Bilinear form for the second and third flows

Now we will construct bilinear representation for systems (31) and (32) to find solutions of MKP-II according to our Proposition 3.1.1. In our paper [11] we applied the Hirota bilinear method to integrate DRD (31). Now we will apply the same method as in the first section to Eq. (32) and MKP-II. As was noticed in Ref. [11], the standard Hirota substitution as the ratio of two functions does not work directly for e^+ and e^-. (This fact also is related with complicated analytical structure of DNLS [10].) To have the standard Hirota substitution, following [11] we first rewrite the systems (31) and (32) in terms of new functions Q^+, Q^-:

$$e^+ = e^{+\int^x Q^+Q^-}Q^+,\ e^- = e^{-\int^x Q^+Q^-}Q^-, \tag{36}$$

and as result we have the systems

$$Q_y^+ = Q_{xx}^+ + Q^+Q^+Q_x^- - \frac{1}{2}(Q^+Q^-)^2Q^+, \tag{37a}$$

$$Q_y^- = -Q_{xx}^- + Q^-Q^-Q_x^+ + \frac{1}{2}(Q^+Q^-)^2Q^-, \tag{37b}$$

and

$$Q_t^+ = Q_{xxx}^+ + 3Q_x^+Q_x^-Q^+ - \frac{3}{2}(Q^+Q^-)^2Q_x^+, \tag{38a}$$

$$Q_t^- = Q_{xxx}^- - 3Q_x^+ Q_x^- Q^- - \frac{3}{2}(Q^+ Q^-)^2 Q_x^-, \tag{38b}$$

Then, due to the fact that

$$Q^+ Q^- = e^+ e^- = U, \tag{39}$$

the systems (37) and (38) provide also solution of MKP-II which we can formulate as below.

Proposition 3.2.1. *Let the functions $Q^+(x, y, t)$ and $Q^-(x, y, t)$, are solutions of the systems* (37) and (38) *simultaneously. Then, the function $U(x, y, t) \equiv Q^+ Q^-$ satisfies the modified Kadomtsev-Petviashvili equation* (*MKP-II*) (34) *or* (35).

To solve the systems (37) and (38) we introduce four real functions g^+, g^-, f^+, f^- according to the formulas

$$Q^+ = \frac{g^+}{f^+}, \quad Q^- = \frac{g^-}{f^-}, \tag{40}$$

or using Eqs. (36) and (39) for the original variables e^+ and e^- we have the following substitution

$$e^+ = \frac{g^+ f^+}{(f^-)^2}, \quad e^- = \frac{g^- f^-}{(f^+)^2}. \tag{41}$$

Then the system (37) bilinearizes in the form

$$(D_y \mp D_x^2)(g^\pm \cdot f^\pm) = 0, \tag{42a}$$

$$D_x^2(f^+ \cdot f^-) + \frac{1}{2} D_x(g^+ \cdot g^-) = 0, \tag{42b}$$

$$D_x(f^+ \cdot f^-) - \frac{1}{2} g^+ g^- = 0. \tag{42c}$$

In a similar way, for the system (38) we have the next bilinear form

$$(D_t - D_x^3)(g^\pm \cdot f^\pm) = 0, \tag{43a}$$

$$D_x^2(f^+ \cdot f^-) + \frac{1}{2} D_x(g^+ \cdot g^-) = 0, \tag{43b}$$

$$D_x(f^+ \cdot f^-) - \frac{1}{2} g^+ g^- = 0. \tag{43c}$$

Comparing these two bilinear forms we can see that the second and the third equations in both systems (42) and (43) are of the same form. This is why for simultaneous solution of both Eqs. (37) and (38) we have the next bilinear system

$$(D_y \mp D_x^2)(g^\pm \cdot f^\pm) = 0, \tag{44a}$$

$$(D_t - D_x^3)(g^\pm \cdot f^\pm) = 0, \tag{44b}$$

$$D_x^2(f^+ \cdot f^-) + \frac{1}{2} D_x(g^+ \cdot g^-) = 0, \tag{44c}$$

$$D_x(f^+ \cdot f^-) - \frac{1}{2} g^+ g^- = 0. \tag{44d}$$

From the last equation we have

$$U = e^+ e^- = Q^+ Q^- = \frac{g^+ g^-}{f^+ f^-} = 2\frac{D_x(f^+ \cdot f^-)}{f^+ f^-} = 2\frac{f_x^+ f^- - f^+ f_x^-}{f^+ f^-}$$

which provides solution of MKP-II by the following formula

$$U = 2\left(\ln \frac{f^+}{f^-}\right)_x \tag{45}$$

3.3. Resonance solitons of MKP-II

Now we consider a solution of the system (44), giving $2+1$ dimensional solution of MKP-II. For one-soliton solution we have

$$g^{\pm} = e^{\eta_1^{\pm}},\ f^{\pm} = 1 + e^{\phi_{11}^{\pm}} e^{\eta_1^{+}+\eta_1^{-}},\ e^{\phi_{11}^{\pm}} = \pm\frac{k_1^{\mp}}{2(k_1^{+}+k_1^{-})^2}, \tag{46}$$

where, $\eta_1^{\pm} = k_1^{\pm}x \pm (k_1^{\pm})^2 y + (k^{\pm})^3 t + \eta_0^{\pm}$. The regularity condition requires $k_1^{+} \leq 0$, $k_1^{-} \geq 0$. Then we have

$$U(x,y,t) = \frac{2k^2}{\sqrt{p^2-k^2}\cosh k(x-py+((k^2+3p^2)/4)t-a_0)+p}, \tag{47}$$

where $k = k_1^{+} + k_1^{-}$, $p = k_1^{-} - k_1^{+} > 0$, and bounded from the below parameter $p^2 > k^2$ is positive $p > 0$. The geometrical meaning of this parameter is $p^{-1} = \tan\alpha$, where α is the slope of the soliton line. Due to the condition $p > 0$, the direction of this line is restricted between $0 < \alpha < \pi/2$. (This is the space analog of the chirality property of dissipaton in $1+1$ dimensions for DNLS [11], when it propagates only in one direction.) The velocity of soliton is two-dimensional vector $\mathbf{v} = (\omega, -\omega/p)$, where $\omega = (k^2+3p^2)/4$, directed at angle γ to the soliton line, where $\cos\gamma = 1 - 1/p^2$. When $p = 1$, the velocity of soliton is orthogonal to the soliton line.

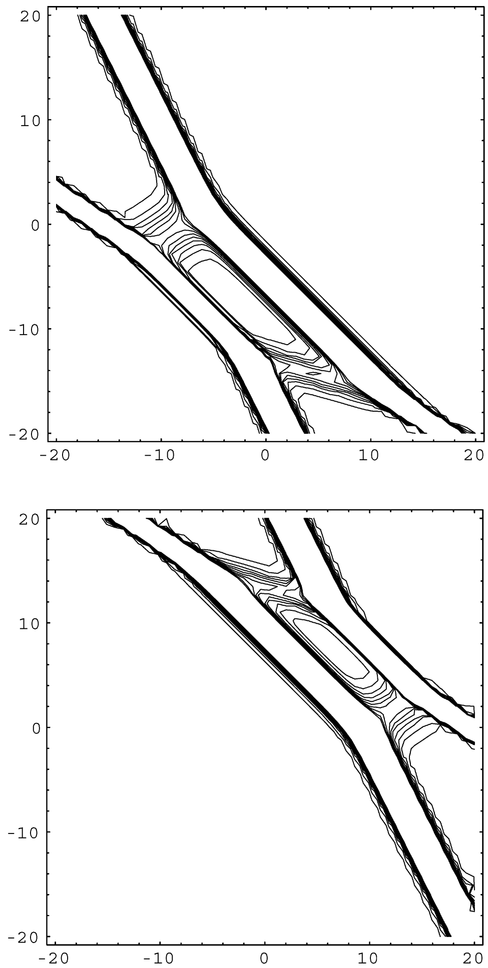

Fig. 1. KP-II four solitons resonance: (a) $t = -5$, (b) $t = 5$.

For two-soliton solution we have

$$g^{\pm} = e^{\eta_1^{\pm}} + e^{\eta_2^{\pm}} + \alpha_1^{\pm} e^{\eta_2^{+}+\eta_2^{-}+\eta_1^{\pm}} + \alpha_2^{\pm} e^{\eta_1^{+}+\eta_1^{-}+\eta_2^{\pm}}, \quad f^{\pm} = 1 + \sum_{i,j=1}^{2} e^{\phi_{ij}^{\pm}} e^{\eta_i^{+}+\eta_j^{-}} + \beta^{\pm} e^{\eta_1^{+}+\eta_1^{-}+\eta_2^{+}+\eta_2^{-}},$$

where $\eta_i^{\pm} = k_i^{\pm} x \pm (k_i^{\pm})^2 y + (k^{\pm})^3 t + \eta_{i0}^{\pm}$, $k_{ij}^{nm} \equiv (k_i^n + k_j^m)$ and

$$\alpha_1^{\pm} = \pm\frac{1}{2}\frac{k_2^{\mp}(k_1^{\pm} - k_2^{\pm})^2}{(k_{22}^{+-})^2(k_{12}^{\pm\mp})^2}, \quad \alpha_2^{\pm} = \pm\frac{1}{2}\frac{k_1^{\mp}(k_1^{\pm} - k_2^{\pm})^2}{(k_{11}^{+-})^2(k_{21}^{\pm\mp})^2}, \quad \beta^{\pm} = \frac{(k_1^{+} - k_2^{+})^2(k_1^{-} - k_2^{-})^2}{4(k_{11}^{+-}k_{12}^{+-}k_{21}^{+-}k_{22}^{+-})^2} k_1^{\mp} k_2^{\mp},$$

$$e^{\phi_{ii}^{\pm}} = \pm\frac{k_i^{\mp}}{2(k_{ii}^{+-})^2}, \quad e^{\phi_{ij}^{+}} = \frac{k_j^{-}}{2(k_{ij}^{+-})^2}, \quad e^{\phi_{ij}^{-}} = -\frac{k_i^{+}}{2(k_{ij}^{+-})^2}.$$

The regularity conditions now are the same as for one-soliton $k_i^{+} \le 0$, $k_i^{-} \ge 0$. Then this solution describes a collision of two solitons propagating in plane and at some value of parameters creating the resonance states (Fig. 2a and b).

4. Conclusions

In the present paper we have constructed virtual soliton resonance solutions for KP-II and MKP-II in terms of dissipative solitons of 1 + 1 dimensional equations as the reaction–diffusion equation and derivative reaction–diffusion

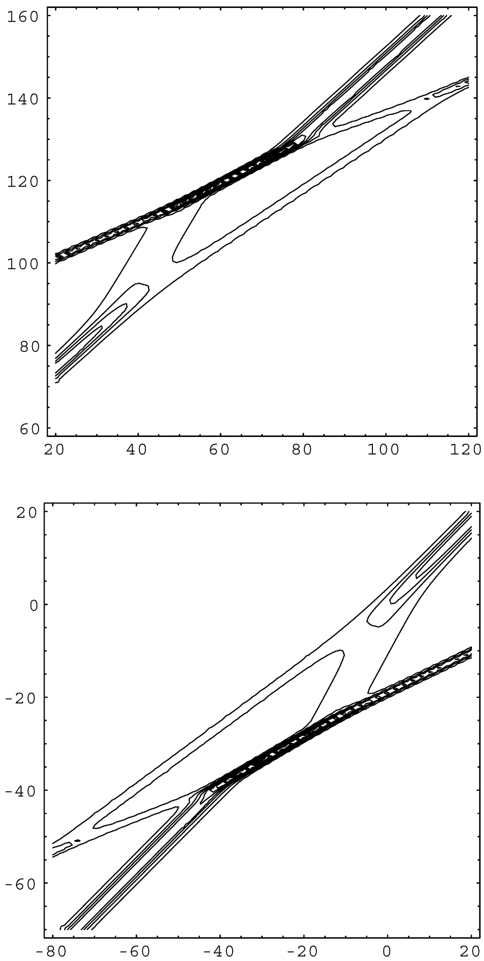

Fig. 2. MKP-II four solitons resonance: (a) $t = 55$, (b) $t = -3$.

equation and their higher members of SL(2,R) AKNS and Kaup-Newell hierarchies. The difference of MKP-II with the KP-II resonance is in the additional restrictions on soliton angles from regularity conditions.

Our results allow to argue that, by combining the Hirota constraints of lower dimensional equations, one may further extend the method in solving higher dimensional completely integrable systems. In more general form, the idea to use couple of equations from the AKNS hierarchy to generate a solution of KP, can be applied also to multidimensional sytems with zero curvature structure such as the Chern-Simons gauge theory. Then our three-dimensional zero curvature representation of KP-II gives flat non-Abelian connection for $SL(2, R)$ and corresponds to a sector of three-dimensional gravity theory.

When this paper has been finished, Konopelchenko attracted our attention to the relations between MKP equation and $1 + 1$ dimensional models by the symmetry reduction of $2 + 1$ dimensional models [9,4]. But in paper [9] a relation of MKP only with Burgers hierarchy has been established. While paper [4] relates MKP with derivative NLS in the Nakamura-Chen form but not in the Kaup-Newell form. Moreover, no results on resonance solitons in those papers are found.

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